Noise-induced phase transitions in a pendulum with a randomly vibrating suspension axis

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The theory of noise-induced phase transitions in a pendulum with a randomly vibrating suspension axis is outlined. It is shown that such transitions are associated with ordering of the system state. The results of numerical simulation of the transitions under consideration are given. The problem of distinguishing noise-induced oscillations and chaotic oscillations of dynamical origin is discussed. [S1063-651X(96)09410-X]

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I. INTRODUCTION

Noise-induced phase transitions, which are akin to those considered below, were studied by Van den Broeck and coworkers in [1,2] for the systems described by finite difference approximations of the partial differential equations of a certain type. In [3] two-dimensional structures induced by noise are obtained numerically. These structures can be considered peculiar turbulence. This fact favors, even if indirectly, the view of one of the authors [4] that turbulence in nonclosed flows is not self-oscillations but is induced by noise.

The transitions in question are of a radically different kind from those which were studied by a number of other researchers (see, for example, [5,6]). In their works the appearance of additional peaks in the probability density under the influence of multiplicative noise, mainly in the systems with multistability, is spoken of as the noise-induced phase transitions. In the case under consideration additional peaks in the probability density do not appear.

We note that we use the term ''nonequilibrium phase transition'' in the same sense as it was used by Haken [7]. Haken called attention to the parallels between phase transitions occurring in systems close to thermodynamic equilibrium state and order-disorder transitions in nonequilibrium systems. In the same sense this term was used in Ref. [8].

As an example of the system in which the noise-induced phase transitions are possible we take a pendulum with a randomly vibrating suspension axis because it is, on the one hand, a very simple system making possible approximate theoretical study, and on the other hand, it is a real physical object.

II. THEORETICAL STUDY OF THE PENDULUM OSCILLATIONS CAUSED BY RANDOM VIBRATION OF ITS SUSPENSION AXIS

The problem of excitations of an oscillator under a parametric random action was first analytically studied by Stratonovich and Romanovsky as early as 1958 [9,10] and later by Dimentberg in 1980 [11]. To obtain the limitation of oscillation amplitude the authors of these works took into account nonlinear friction. In fact, the inclusion of nonlinear friction is not necessary since the limitation of amplitude can occur owing to nonlinearity of the restoring force. However, the inclusion of nonlinear friction makes it possible to obviate random rotations of the pendulum through an angle divisible by 2π. These rotations make the analysis of the obtained results more difficult.

The motion equation for a pendulum with a randomly vibrating suspension axis with regard to nonlinear friction can be written as

$$\ddot{\varphi}+2\beta(1+\alpha\dot{\varphi}^2)\dot{\varphi}+\omega_0^2[1+\xi(t)]\sin\varphi=0, \qquad (1)$$

where φ is the pendulum angular deviation from the equilibrium position, $\omega_0=\sqrt{mbg/J}$ is the natural frequency of small free pendulum's oscillations, J and m are the moment of inertia and the mass of the pendulum, b is the distance between the center of mass and the suspension axis, g is the acceleration of gravity, $\beta=H/2J$ is the damping factor, $H\dot{\varphi}$ is the moment of the friction force in the linear approximation, α is the coefficient of nonlinear friction, and $\xi(t)$ is the acceleration of the suspension axis that is a comparatively wideband random process with nonzero power spectrum density at the frequency $2\omega_0$. We assume that the intensity of the suspension axis' vibrations is moderately small, so that the pendulum's oscillations can be considered small to an extent that $\sin\varphi$ can be presented as

$$\sin\varphi\approx\left(1-\frac{1}{6}\varphi^2\right)\varphi. \qquad (2)$$

For generality below we consider the nonlinearity of the form $(1-\gamma\varphi^2)\varphi$, which coincides with (2) for $\gamma=1/6$.

An approximate analytical solution of the problem can be obtained on the assumptions that $\beta/\omega_0\sim\epsilon$, $\gamma\varphi^2\sim\epsilon$, $\xi(t)\sim\sqrt{\epsilon}$, where ϵ is a certain small parameter which should be put equal to unity in the final results. With these assumptions Eq. (1) in view of (2) is conveniently written as

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$$\ddot{\varphi}+\omega_0^2\varphi=\epsilon[-2\beta(1+\alpha\dot{\varphi}^2)\dot{\varphi}+\omega_0^2\gamma\varphi^3] -\sqrt{\epsilon}\xi(t)(1-\epsilon\gamma\varphi^2)\varphi. \quad (3)$$

Equation (3) can be solved by the Krylov-Bogolubov method with a precision to the second approximation with respect to the small parameter ϵ; to do this we set $\varphi=A\cos\psi+\epsilon u_1+\epsilon^2 u_2+\cdots$, where $\psi=\omega_0 t+\phi$,

$$\dot{A}=\epsilon f_1+\epsilon^2 f_2+\cdots,$$
$$\dot{\phi}=\epsilon F_1+\epsilon^2 F_2+\cdots, \quad (4)$$

$u_1,u_2,\cdots,f_1,f_2,\cdots,F_1,F_2,\cdots$, are unknown functions. By using the Krylov-Bogolubov technique for stochastic equations (see [10,12]) we find the expressions for the unknown functions f_1, f_2, and F_1 (the function F_2 gives only small additions to the function F_1 and so is of no interest). Retaining only the nonlinear term in the function f_2 and substituting the expressions found into Eqs. (4) we obtain

$$\dot{A}=\left[-\beta\left(1+\frac{3}{4}(\alpha\omega_0^2+\gamma)A^2\right)+\frac{\omega_0}{2}\overline{\xi\sin 2\psi}\right]A, \quad (5)$$

$$\dot{\phi}=-\frac{3}{8}\omega_0\gamma A^2+\omega_0\overline{\xi\cos^2\psi}, \quad (6)$$

where the bars over the expressions signify time averaging. As indicated in [10], in Eq. (5) we have

$$\overline{\xi\sin 2\psi}=\langle\overline{\xi\sin 2\psi}\rangle+\zeta_1(t), \quad (7)$$

where the angular brackets signify averaging over statistical ensemble, $\zeta_1(t)$ is a random process that can be considered as white noise with zero mean value, and the intensity

$$K_1=\frac{1}{2}\kappa(2\omega_0), \quad (8)$$

where

$$\kappa(2\omega_0)=\int_{-\infty}^{\infty}\langle\xi(t)\xi(t+\tau)\rangle\cos 2\omega_0\tau\, d\tau$$

is the power spectrum density of the process $\xi(t)$ at the frequency $2\omega_0$. In the expression (7) the value $\langle\overline{\xi\sin 2\psi}\rangle$ is different from zero owing to the correlation between ξ and ϕ; it is equal to

$$\langle\overline{\xi\sin 2\psi}\rangle=\frac{\omega_0}{4}\kappa(2\omega_0)=\frac{\omega_0}{2}K_1. \quad (9)$$

In a similar manner, in Eq. (6) we have

$$\overline{\xi\cos^2\psi}=\langle\overline{\xi\cos^2\psi}\rangle+\zeta_2(t), \quad (10)$$

where

$$\langle\overline{\xi\cos^2\psi}\rangle=\frac{\omega_0}{4}\int_{-\infty}^{0}\langle\xi(t)\xi(t+\tau)\rangle\sin 2\omega_0\tau\, d\tau\equiv M. \quad (11)$$

$\zeta_2(t)$, much like $\zeta_1(t)$, can be considered as white noise with zero mean value and the intensity

$$K_2=\frac{1}{4}\left(\kappa(0)+\frac{1}{2}\kappa(2\omega_0)\right). \quad (12)$$

The value of M depends on the characteristics of the random process $\xi(t)$: if $\xi(t)$ is white noise then $M=0$, but if $\xi(t)$ has a finite correlation time, for example, as its power spectrum density is

$$\kappa(\omega)=\frac{\alpha^2\kappa(2\omega_0)}{(\omega-2\omega_0)^2+\alpha^2},$$

then

$$M=-\frac{\alpha\omega_0^2\kappa(2\omega_0)}{4(16\omega_0^2+\alpha^2)}.$$

In view of (7)–(12) we rewrite Eqs. (5) and (6) as

$$\dot{A}=\left(\frac{\omega_0^2}{4}K_1-\beta-\frac{3}{4}\beta\tilde{\gamma}A^2\right)A+\frac{\omega_0}{2}A\zeta_1(t),$$
$$\dot{\phi}=\omega_0 M-\frac{3}{8}\omega_0\gamma A^2+\omega_0\zeta_2(t), \quad (13)$$

where $\tilde{\gamma}=\gamma+\alpha\omega_0^2$.

The Fokker-Planck equation associated with Eqs. (13) is

$$\frac{\partial w(A,\phi)}{\partial t}=-\frac{\partial}{\partial A}\left[\left(\frac{\omega_0^2 K_1}{4}\eta-\frac{3}{4}\beta\tilde{\gamma}A^2\right)Aw(A,\phi)\right] -\omega_0\left(\frac{3}{8}\gamma A^2-M\right)\frac{\partial w(A,\phi)}{\partial\phi} +\frac{K_1\omega_0^2}{8}\frac{\partial^2}{\partial A^2}[A^2 w(A,\phi)]+\frac{K_2\omega_0^2}{2}\frac{\partial^2 w(A,\phi)}{\partial\phi^2}, \quad (14)$$

where

$$\eta=1-\frac{4\beta}{\omega_0^2 K_1}.$$

As will be seen from the following, the parameter η characterizes the extent to which the noise intensity is in excess of its critical value.

The steady-state solution of Eq. (14), satisfying the condition for the probability flux to be equal to zero, is independent of ϕ. It is conveniently written as

$$w(A,\phi)=\frac{C}{2\pi A^2}\exp\left\{\int_0^A\left(\frac{2\eta}{A}-\frac{6\beta\tilde{\gamma}}{\omega_0^2 K_1}A\right)dA-\int_0^1\frac{2\eta}{A}dA\right\}. \quad (15)$$

The constant C is determined from the normalization condition

$$\int_0^{2\pi}\int_0^{\infty}w(A,\phi)A\; dA\; d\phi=1.$$

Upon integrating (15) with respect to ϕ and calculating the integral under the exponential symbol, we find the expression for the probability density of the amplitude of oscillations:

$$w(A)=CA^{2\eta-1}\exp\left\{-\frac{3\beta\tilde{\gamma}}{\omega_0^2K_1}A^2\right\}. \quad (16)$$

From the normalization condition we obtain

$$C=2\begin{cases}\left(\frac{\omega_0^2K_1}{3\beta\tilde{\gamma}}\right)^{-\eta}\frac{1}{\Gamma(\eta)} & \text{for } \eta\geqslant 0\\ 0 & \text{for } \eta\leqslant 0.\end{cases}$$

Hence,

$$w(A)=2\begin{cases}\left(\frac{\omega_0^2K_1}{3\beta\tilde{\gamma}}\right)^{-\eta}\frac{1}{\Gamma(\eta)}A^{2\eta-1}\exp\left\{-\frac{3\beta\tilde{\gamma}}{\omega_0^2K_1}A^2\right\} & \text{for } \eta\geqslant 0\\ \delta(A) & \text{for } \eta\leqslant 0.\end{cases} \quad (17)$$

We note that with the increase of the noise intensity the parameter η changes the sign and one has the transition from δ function to the normalizable probability distribution. The fact that for $\eta\leqslant 0$ the probability density of the amplitude turns out to be a δ function is associated with neglect of additive noise. (Consideration of the effect of additive noise is conducted by Dimentberg [11].)

Using (17) we can find $\langle A\rangle$ and $\langle A^2\rangle$. They are

$$\langle A\rangle=\begin{cases}\sqrt{\frac{\omega_0^2K_1}{3\beta\tilde{\gamma}}}\frac{\Gamma(\eta+1/2)}{\Gamma(\eta+1)}\eta & \text{for } \eta\geqslant 0\\ 0 & \text{for } \eta\leqslant 0,\end{cases}$$

$$\langle A^2\rangle=\begin{cases}\frac{\omega_0^2K_1}{3\beta\tilde{\gamma}}\eta & \text{for } \eta\geqslant 0\\ 0 & \text{for } \eta\leqslant 0.\end{cases} \quad (18)$$

It is seen from this that for $\eta>0$ the parametric excitation of the pendulum's oscillations occur under the effect of noise. This manifests itself in the fact that the mean values of the amplitude and the amplitude squared become different from zero. If an observer detects such oscillations and does not know the causes for their occurrence then he can draw the conclusion that he views chaotic self-oscillations. The question naturally arises whether or not one can distinguish between the process observed and chaotic self-oscillations. This problem will be discussed below.

III. RESULTS OF NUMERICAL SIMULATION OF THE OSCILLATIONS OF A PENDULUM WITH A RANDOMLY VIBRATING SUSPENSION AXIS

Because the theoretical results obtained are approximate and give no way of determining the pendulum's oscillation shape, we have studied numerically solutions of the equation

$$\ddot{\varphi}+0.2(1+\alpha\dot{\varphi}^2)\dot{\varphi}+[1+\xi(t)]\sin\varphi=0, \quad (19)$$

where $\xi(t)$ is sufficiently wideband noise whose power spectrum is presented in Fig 1. The term describing nonlinear friction is included in Eq. (19) to avoid the pendulum's rotations as the noise intensity is essentially in excess of its critical value. The study has shown that, as the noise intensity increases, the mean values of the instantaneous amplitude and the instantaneous amplitude squared of the pendulum's oscillations increase from zero at $\kappa(2)$ equal to $\kappa_{cr}(2)=0.8$ onwards. [The instantaneous amplitude can be calculated by using the Hilbert transform (see, for example, [13,14]).] The corresponding dependencies are shown in Fig. 2. We see that at initial parts these dependencies agree closely with the theoretical dependencies determined by the formulas (18) that are calculated by us in the assumption that the noise intensity is near its critical value.

If the noise intensity differs little from its critical value then the oscillations excited (see Fig. 3) closely resemble chaotic self-oscillations coming into existence as a result of the stability's loss of an equilibrium position through merging with an unstable limit cycle and, therefore, possessing the property of intermittency [8]. We emphasize that turbulence for transient Reynolds numbers also exhibits this property [15–17]. It is no chance that the first theoretical works concerning the intermittency phenomenon were made by the specialists in the field of turbulence [18].

As the noise intensity increases the duration of the regions where the pendulum oscillates in the immediate vicinity of the equilibrium position is progressively reduced, and eventually the regions die out. This is illustrated in Fig. 4.

Inasmuch as the pendulum's oscillations under consideration are caused by nothing but the noise, their dimension

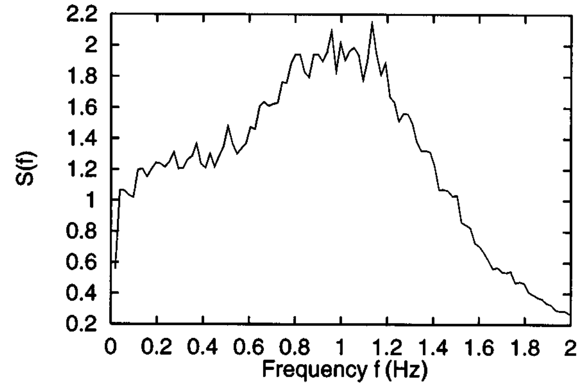

FIG. 1. The power spectrum of the noise $\xi(t)$.

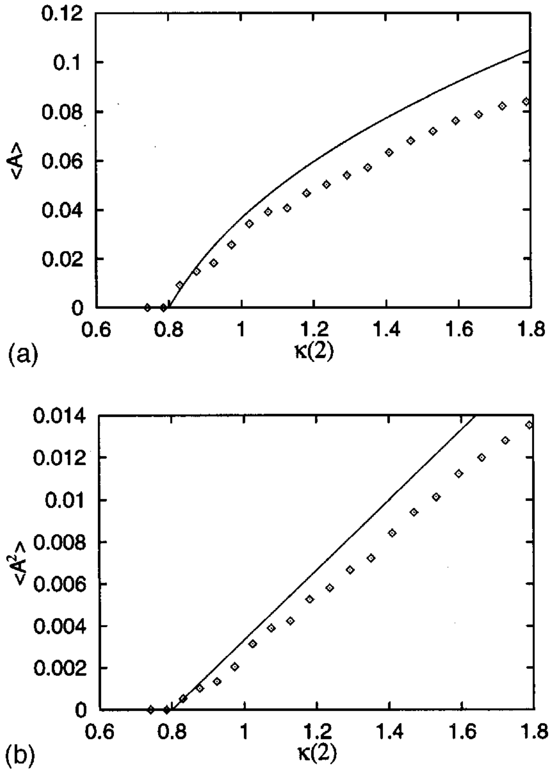

FIG. 2. The dependencies of $\langle A\rangle$ (a) and $\langle A^2\rangle$ (b) on the noise spectral constituent $\kappa(2)$. The corresponding theoretical dependencies calculated by the formulas (18) are shown by dashed lines.

would be expected to be sufficiently large. However, the calculations of correlation dimension, performed by us both in ordinary Takens' space and with using the Karhunen-Loeve and well-adapted bases [19], have shown that the dimension is not large. The saturation of the correlation dimension with increasing embedding space dimension points to this. (It should be noted that the corresponding correlation integrals have no clearly defined linear part, making the exact evaluation of the dimension difficult.)

An example of the dependence of the correlation dimension on the embedding space dimension is shown in Fig. 5. As the noise intensity increases the dimension increases only slightly, but it remains finite. The dependence of the correlation dimension ν on the relative spectrum density $\kappa(2)/\kappa_{cr}(2)$ is depicted in Fig. 6. So, the dimension gives no way of distinguishing between noise-induced oscillations and chaotic oscillations of dynamical origin. An example of such oscillations will be considered below. It should be particularly emphasized that the result obtained is in contradiction with popular opinion that the dimension is precisely the characteristic which allows the chaotic oscillations in dynamical systems and random oscillations caused by noise to be distinguishable. True, in the past few years there have appeared several papers [20,21] in which it is shown that a time series with a $1/f^{\alpha}$ power spectrum can exhibit a finite correlation dimension (at least, for $1\leqslant\alpha\leqslant3$). However, as we shall see subsequently, the power spectrum of the noise-induced oscillations observed by us is not always $1/f^{\alpha}$; nevertheless, the dimension is finite.

In view of the fact that the dimension corresponding to noise-induced pendulum oscillations is finite we can assert

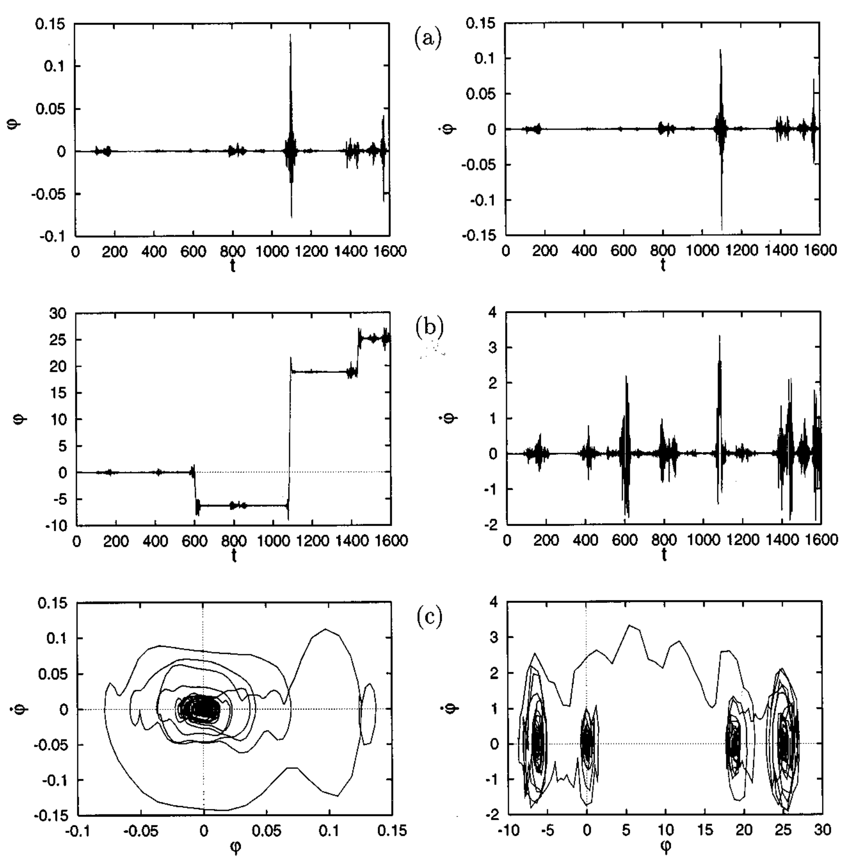

FIG. 3. The dependencies $\varphi(t)$ and $\dot{\varphi}(t)$ for $\alpha=0$, (a) $\kappa(2)/\kappa_{cr}(2)=1.01$ and (b) $\kappa(2)/\kappa_{cr}(2)=1.06$; (c) the projections of the corresponding phase portraits on the plane $\varphi(t),\dot{\varphi}(t)$.

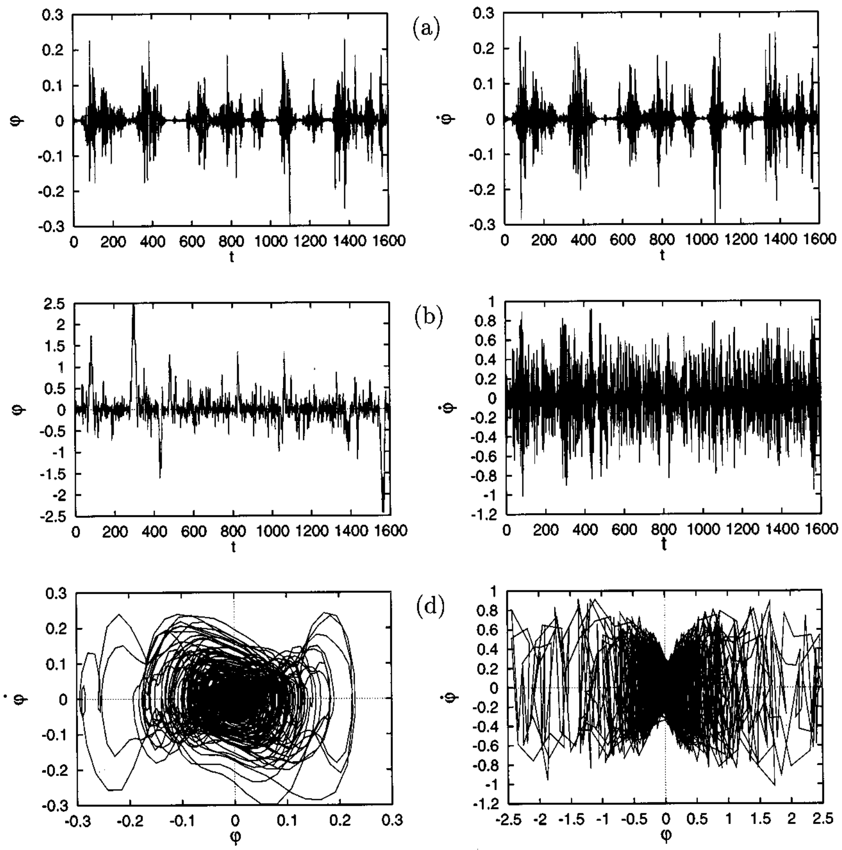

FIG. 4. The dependencies $\varphi(t)$, $\dot{\varphi}(t)$ for $\alpha=100$, (a) $\kappa(2)/\kappa_{cr}(2)=1.25$ and (b) $\kappa(2)/\kappa_{cr}(2)=14$; the projections of the corresponding phase portraits on the plane $\varphi(t),\dot{\varphi}(t)$ (c).

that an attractor is induced in a certain phase space associated with the pendulum motion, e.g., in Takens' space.

For comparison, let us consider chaotic pendulum oscillations caused by sufficiently large periodic vibration of its suspension axis. Taking into account nonlinear friction we write the equations of these oscillations as

$$\ddot{\varphi}+2\beta(1+\alpha\dot{\varphi}^2)\dot{\varphi}+(1+B\cos 2t)\sin\varphi=0, \tag{20}$$

where B is the relative amplitude of the suspension axis acceleration. The behavior of the solution of Eq. (20) under changes of the parameter B for $\alpha=0$ was studied in detail in [22] by means of computer simulation. We have repeated the calculations performed by McLaughlin for a number of values of B for which the pendulum oscillations are chaotic. An example of such oscillations is represented in Fig. 7(a). It is seen from this figure that the pendulum rotates irregularly in one or another direction. This causes a considerable slow drift of the angle φ. The nonlinear friction, if it is of a sufficient value, results in cessation of rotation and oscillations of the pendulum about its equilibrium position [see Fig. 7(b)]. The correlation dimension of the attractor associated with these chaotic pendulum oscillations is equal to 2.51 ± 0.05 for $B=3$, $\alpha=0$ and 2.09 ± 0.03 for $B=3.5$,

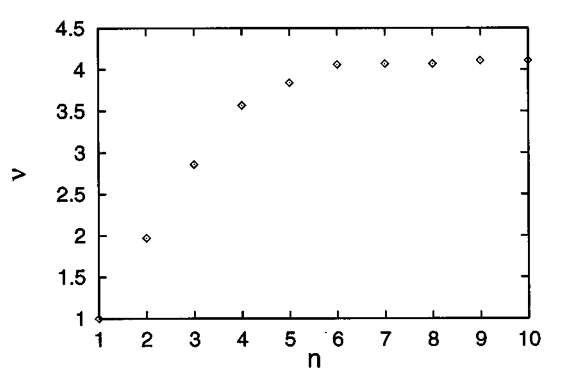

FIG. 5. The dependence of the correlation dimension ν on the embedding space dimension n for $\kappa(2)/\kappa_{cr}(2)=14$, $\alpha=100$.

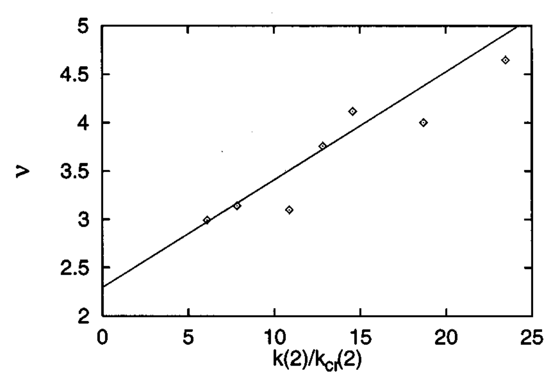

FIG. 6. The dependence of the correlation dimension ν on the relative spectrum density $\kappa(2)/\kappa_{cr}(2)$. We see that the points calculated are located in the vicinity of a straight line which is drawn in the figure by minimizing the mean square error (solid line).

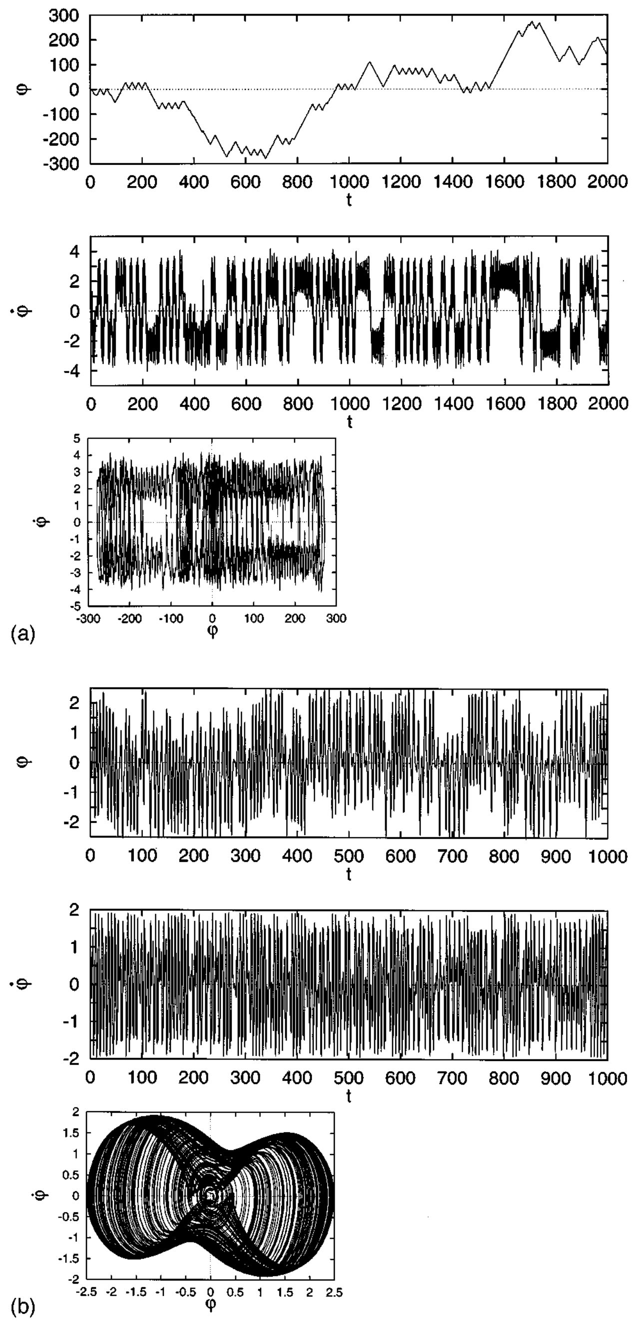

FIG. 7. The solution of Eq. (20) and the projections of phase portraits on the plane $\varphi,\dot{\varphi}$ for $B=3$, $\alpha=0$ (a), $B=3.5$, $\alpha=2$ (b).

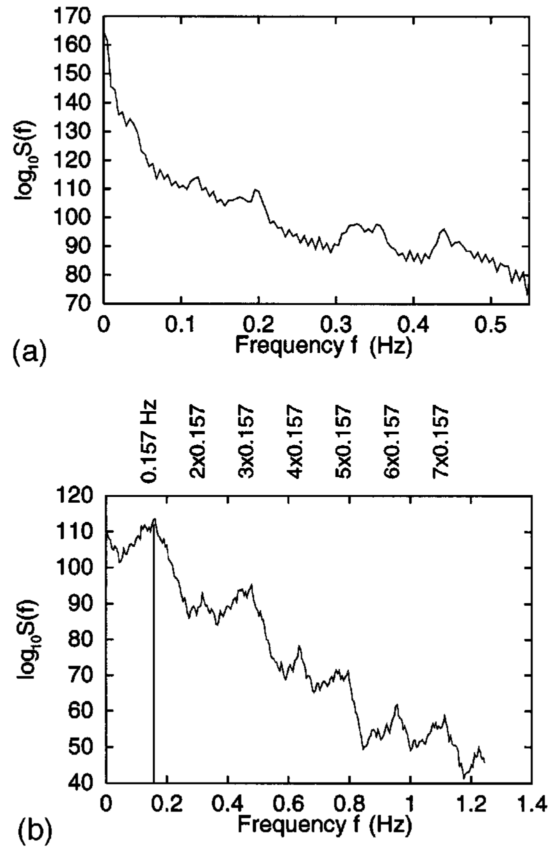

FIG. 8. The power spectra for the solutions of Eq. (20) for $B=3$, $\alpha=0$ (a) and $B=3.5$, $\alpha=2$ (b).

$\alpha=2$. We see that the presence of nonlinear friction results in significant decrease of the dimension.

Of considerable interest are the power spectra of the oscillations excited. In the case where the pendulum is excited by a harmonic vibration of the suspension axis and nonlinear friction is negligible, its power spectrum contains low-frequency part caused by the slow drift of φ; the power spectrum density decreases with increasing frequency but not monotonically [see Fig. 8(a)]. With nonlinear friction the low-frequency part of the power spectrum substantially decreases and distinct peaks at the frequencies multiple to natural come into existence [Fig. 8(b)].

In the case where the pendulum is excited by noise and nonlinear friction is negligible, its power spectrum has a peak close to the natural frequency if the noise intensity differs little from its critical value [Fig. 9(a)]. As the noise intensity increases this peak decreases and disappears eventually; the spectrum becomes monotonically descending [Fig. 9(b)] and reminiscent of the flicker noise spectrum. For large noise intensities the spectrum can be approximated by an exponential dependence of the form $1/f^n$, where $n=12$ for $\kappa(2)/\kappa_{cr}(2)=22$ [Fig. 9(c)]. With nonlinear friction the qualitative behavior of the power spectrum is the same [see Figs. 9(d)–9(f)]. It differs only in the form of the approximation for sufficiently large noise intensities [Fig. 9(g)]. The correlation functions for the parameters corresponding to Figs. 9(d) and 9(f) are shown in Fig. 10. We see that the correlation time is not too large, and therefore, the contention that the dimension of $1/f^\alpha$ noise is finite owing to a large correlation time [20,21] is not appropriate for our case.

IV. RYTOV-DIMENTBERG CRITERION

Let us revert to the question of whether or not one can distinguish between noise-induced oscillations and chaotic oscillations of dynamical origin. A similar question was first formulated by Rytov [13] and later by Dimentberg [11], as applied to the problem of distinguishing between noise passed through a linear narrow-band filter and periodic but noisy self-oscillations. It was shown that in the case of noisy self-oscillations the probability density for instantaneous amplitude squared has to peak at a certain finite value of the amplitude, whereas for noise passed through a filter it has to be monotonically decreasing. In the case of chaotic oscilla-

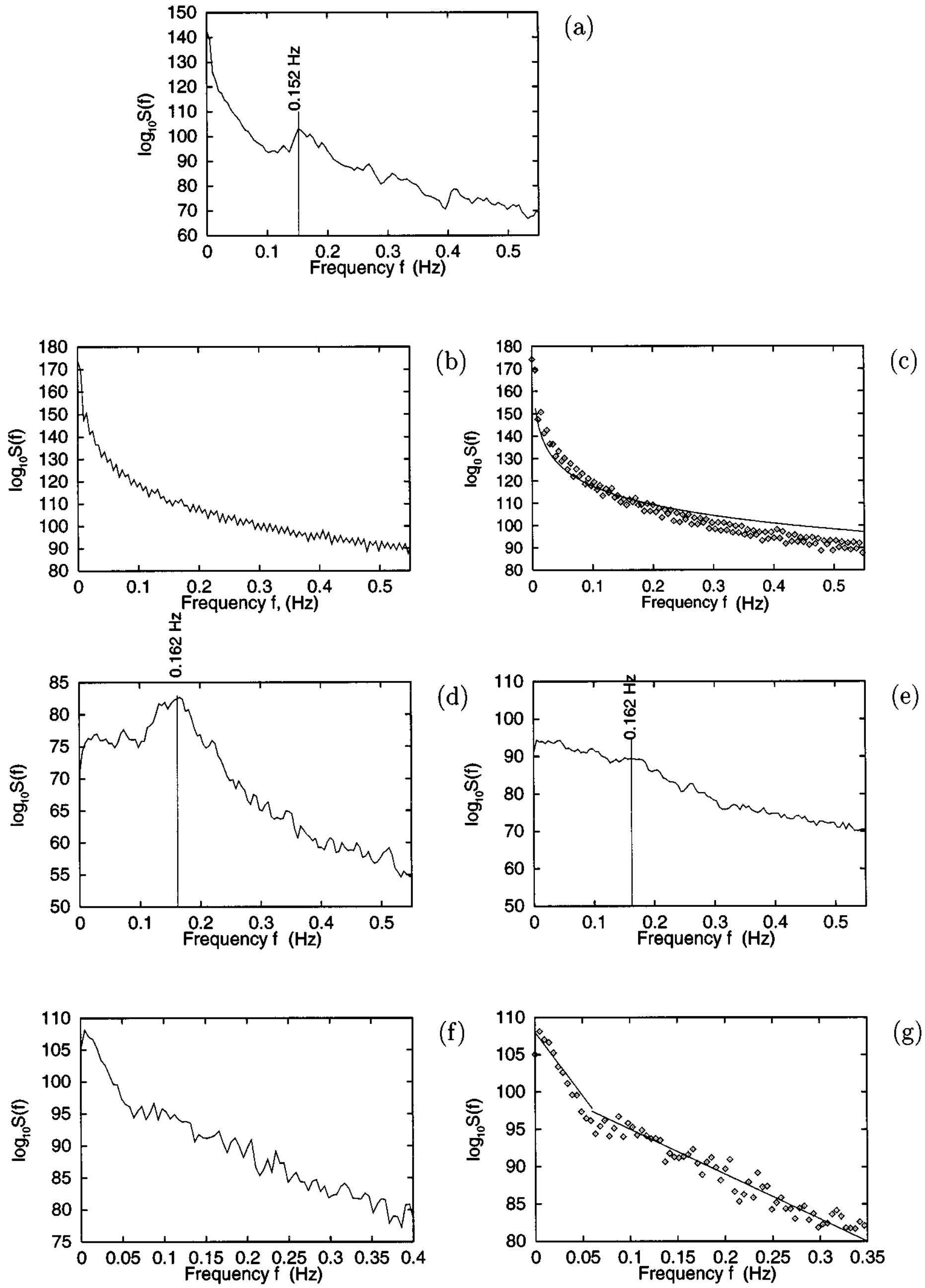

FIG. 9. The power spectra for the solutions of Eq. (19) for $\alpha=0$, (a) $\kappa(2)/\kappa_{cr}(2)=1.06$ and (b) $\kappa(2)/\kappa_{cr}(2)=22$; $\alpha=100$, (d) $\kappa(2)/\kappa_{cr}(2)=1.25$, (e) $\kappa(2)/\kappa_{cr}(2)=4.6$ and (f) $\kappa(2)/\kappa_{cr}(2)=14$. The approximations of the power spectrum logarithm by $90-12\ln f$ for $\alpha=0$, $\kappa(2)/\kappa_{cr}(2)=22$ (c) and by the intercepts of two straight lines ($108-170f$ for $f\leqslant 0.06$ and $101-60f$ for $f\geqslant 0.06$) for $\alpha=100$, $\kappa(2)/\kappa_{cr}(2)=14$ (g).

tions of dynamical origin the probability density for instantaneous amplitude squared would be also expected to peak at one or several values of the amplitude. We have verified this statement by an example of chaotic pendulum oscillations caused by periodic vibration of the suspension axis. The corresponding histogram for the probability density of instantaneous amplitude is shown in Fig. 11. We see that the probability density is not monotonically decreasing with increasing amplitude but has several peaks only slightly defined.

It follows from the results presented in the first section that in the case of the parametric excitation of the pendulum's oscillations under the effect of random vibration of the suspension axis, the probability density for the value $x=\tilde{\gamma}A^2$ is $\tilde{w}(x)=w(\sqrt{x/\tilde{\gamma}})/2\sqrt{\tilde{\gamma}x}$, where $w(\sqrt{x/\tilde{\gamma}})$ is determined by the expression (17). The dependence $\tilde{w}(x)$ for $\eta=0.2$ is shown in Fig. 12(a). We see that the probability density for amplitude squared, calculated analytically, is monotonically decreasing with increasing amplitude. The similar results are also obtained from data of numerical simulation. The histograms of the probability density for A^2 calculated from the numerical solution of Eq. (19) for $\kappa(2)/\kappa_{cr}(2)=1.25$ and $\kappa(2)/\kappa_{cr}(2)=14$ are represented in Figs. 12(b) and 12(c). Dimentberg suggested also another version of this criterion. In place of instantaneous amplitude, the probability density for the process $x(t)$ in itself is analyzed. It is shown that if the probability density for $x>0$ is not monotonically decreasing then the process $x(t)$ is self-oscillatory. But if the probability density for $x>0$ is monotonically decreasing then the process $x(t)$ can be both self-oscillatory and noise passed through a filter. Although the author passes over in silence this fact it is evident that for using this criterion the probability density for x should be an even function.

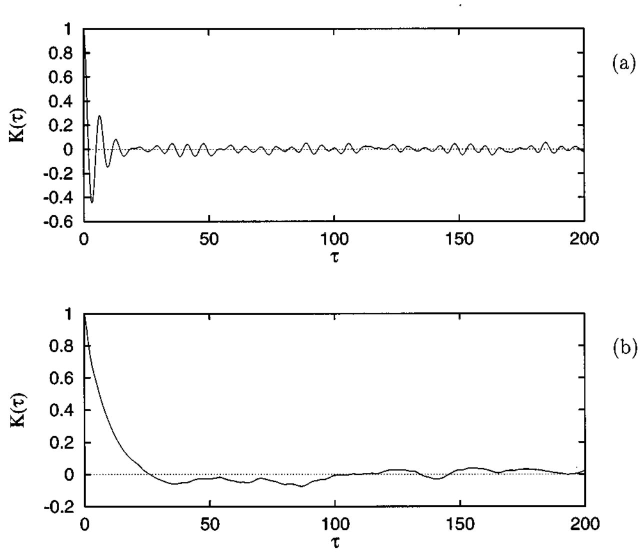

FIG. 10. The correlation functions for the solutions of Eq. (19) for $\alpha = 100$, (a) $\kappa(2)/\kappa_{cr}(2) = 1.25$ and (b) $\kappa(2)/\kappa_{cr}(2) = 14$.

We have verified the second version of the Dimentberg criterion for both noise-induced pendulum's oscillations and chaotic oscillations caused by harmonic action. We have detected that this version is also usable. Thus, in spite of the essentially nonlinear transformation of noise, in the case under consideration the Rytov-Dimentberg criterion is true. It is undeniable that the question of the verity of this criterion in the general case is still an open question.

V. EXCITATION OF OSCILLATIONS OF A PENDULUM WITH RANDOMLY VIBRATING SUSPENSION AXIS AS A NOISE-INDUCED PHASE TRANSITION. THE KLIMONTOVICH CRITERION

The excitation of pendulum oscillations at the sacrifice of noise parametric action can be treated as the occurrence at $\eta = 0$ of a nonequilibrium phase transition of the second kind. One of the values $\langle A \rangle$ and $\langle A^2 \rangle$ can be taken as a parameter of order characterizing this transition. It follows from (18) and Fig. 2 that in deciding on such a parameter of order, the critical index is equal to unity.

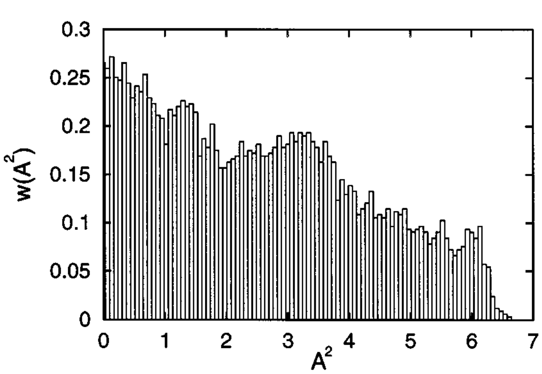

FIG. 11. The histogram for the probability density of instantaneous amplitude squared for $B = 3.5$, $\alpha = 2$.

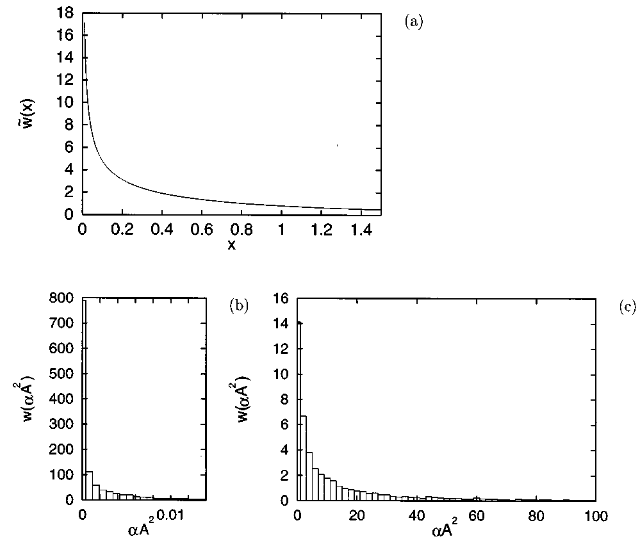

FIG. 12. The theoretical dependence $\widetilde{w}(x) = (2\widetilde{\gamma}^{\eta}/C)w(x)$ for $\eta = 0.2$ (a) and the histograms of the probability density for A^2 calculated from the numerical solution of Eq. (19) for $\kappa(2)/\kappa_{cr}(2) = 1.25$ (b) and $\kappa(2)/\kappa_{cr}(2) = 14$ (c).

In order to make certain that the motion in the system becomes more ordered after the transition considered we use a criterion suggested by Klimontovich [23–25]. This criterion consists of the following. Two states are chosen for the system under study which correspond to two different values of a certain controlling parameter a. One of these states corresponding to $a = a_0$ is arbitrarily taken by us as the state of *physical chaos*. Let the probability density for the set of variables X describing the system's state be symbolized by $w(X,a)$. We represent $w(X,a_0) \equiv w_0(X)$ as Gibbs' canonical distribution

$$w_0(X) = \exp\left\{ \frac{F_0(D_0) - H(X,a_0)}{D_0} \right\}, \qquad (21)$$

where F_0 is the free energy, $H(X,a_0)$ is the Hamilton function, D_0 is the temperature in appropriate units. Let us denote

$$\frac{F_0(D_0) - H(X,a_0)}{D_0} = -H_{ef}$$

and take H_{ef} as the effective Hamilton function which is independent of the parameter a. It is evident that the mean value of the effective Hamilton function, which is equal to the effective energy, depends generally on a. Klimontovich proposes to renormalize the initial probability distribution so that the effective energy in the initial (for $a = a_0$) and the final (for $a = a_0 + \Delta a$) states are coincident. For this purpose the renormalized probability density $\widetilde{w}_0(X,a,\Delta a)$ is entered which satisfies the condition

$$\int H_{ef}\widetilde{w}_0(X,a,\Delta a)dX = \int H_{ef} w(X,a_0 + \Delta a)dX. \qquad (22)$$

In Eq. (22) the probability density $\widetilde{w}_0(X,a,\Delta a)$, as well as $w_0(X)$, can be represented in terms of Gibbs' canonical distribution

$$\widetilde{w}_0(X,a,\Delta a)=\exp\left\{\frac{F(D)-H_{ef}}{D}\right\}, \tag{23}$$

where $F(D)$ is the effective free energy and D is the effective temperature depending on Δa. The unknown function $F(D)$ is determined from the normalization condition

$$\int \widetilde{w}_0(X,a,\Delta a)dX=1, \tag{24}$$

whereas the dependence of D on Δa is found from Eq. (22). Comparing (21) with (23) we see that $D(0)=1$ and $F(1)=0$.

According to Klimontovich's criterion, if the value of $D(\Delta a)$ found is more than unity then the state of the system corresponding to $a=a_0+\Delta a$ is more ordered than the state corresponding to $a=a_0$; i.e., in this case the initial state is properly taken by us as the state of physical chaos. [The aforesaid is valid if in passing from a to $a-\Delta a$ the value of D is found to be less than unity; otherwise the procedure is more complicated (see [23–25]). We shall assume that this simplest situation takes place.] Klimontovich proposes to use the difference in the entropies $\widetilde{S}_0=-\int\widetilde{w}_0(X,a,\Delta a)\ln\widetilde{w}_0(X,a,\Delta a)dX$ and $S=-\int w(X,a_0+\Delta a)\ln w(X,a_0+\Delta a)dX$ as a quantitative estimate of the extent to which the state of the system becomes more ordered as a changes from a_0 to $a_0+\Delta a$. It follows from the normalization condition and Eq. (22) that

$$\Delta S=\widetilde{S}_0-S=\int w\ln\frac{w}{\widetilde{w}_0}dX. \tag{25}$$

We note that the value of ΔS, which is determined by the expression (25), cannot be negative, even though we would choose the state of physical chaos improperly, i.e., the value of D would be found to be less than unity. The reason is that $\ln x\geqslant 1-1/x$; the latter follows from the integral representation of logarithm.

Let us revert to our problem and take the state corresponding to $\eta=\eta_0$ as the state of physical chaos, and we take the state corresponding to $\eta>\eta_0$ as the state for which we want to determine the extent to which it is ordered. Setting $\eta_0\ll 1$ and performing the calculations indicated above, we find that $D=1+2(\eta-\eta_0)(1+2\eta+3\eta^2+\cdots)$; i.e., the state of physical chaos was taken by us properly. The calculation of the difference in the entropies δS is too cumbersome, but it can be shown that $\delta S\sim\eta^2(\eta-\eta_0)$. We emphasize that the expressions found are true for $\eta_0=0$ as well. So, we have obtained that in the transition considered above the state of the system becomes more ordered from the Klimontovich criterion standpoint.

VI. STABILIZATION OF THE UPPER EQUILIBRIUM POSITION OF A PENDULUM WITH A RANDOMLY VIBRATING SUSPENSION AXIS

It is well known that the upper equilibrium position of a pendulum with a harmonically vibrating suspension axis can become stable if the frequency of the vibration is sufficiently high (see, for example, [26–28]). This phenomenon was observed experimentally by Kapitsa [29,30]. Below it is shown that the similar phenomenon can also be observed in the case of random, but sufficiently high-frequency, vibration of pendulum's suspension axis. To do this we consider the equation

$$\ddot{\varphi}+2\beta\dot{\varphi}+[1+\xi(t)]\sin\varphi=0. \tag{26}$$

If the power spectrum of the random process $\xi(t)$ peaks at a sufficiently high frequency, then the deviations of the variable φ caused by the random vibration of the suspension axis are small. Setting $\varphi=\langle\varphi\rangle+\delta\varphi$, where $\delta\varphi\ll\langle\varphi\rangle$ in (26), we obtain

$$\langle\ddot{\varphi}\rangle+2\beta\langle\dot{\varphi}\rangle+\sin\langle\varphi\rangle+\cos\langle\varphi\rangle\langle\xi(t)\delta\varphi\rangle=0, \tag{27}$$

$$\delta\ddot{\varphi}+2\beta\delta\dot{\varphi}+\cos\langle\varphi\rangle\delta\varphi+\xi(t)\sin\langle\varphi\rangle=0. \tag{28}$$

Let us find an approximate solution of Eqs. (27) and (28) in the vicinity of the pendulum's upper equilibrium position when $\cos\langle\varphi\rangle$ is close to -1. A steady solution of Eq. (28) for $\beta\ll 1$ and $\cos\langle\varphi\rangle\approx-1$ is

$$\delta\varphi(t)=-\frac{1}{2}\int_{-\infty}^{t}(e^{t-t'}-e^{-(t-t')})\xi(t')\sin\langle\varphi(t')\rangle dt'. \tag{29}$$

We find herefrom that

$$\langle\xi(t)\delta\varphi\rangle=-\frac{1}{2}\int_{-\infty}^{t}(e^{t-t'}-e^{-(t-t')})\times\langle\xi(t')\xi(t)\rangle\sin\langle\varphi(t')\rangle dt'. \tag{30}$$

Substituting $t'-t=\tau$ in this expression and taking into account that the value $\langle\varphi\rangle$ does not considerably change during a correlation time of the random process $\xi(t)$, we rewrite (30) in the following form:

$$\langle\xi(t)\delta\varphi\rangle=\frac{1}{2}\sin\langle\varphi(t)\rangle\int_0^{\infty}(e^{-\tau}-e^{\tau})\langle\xi(t)\xi(t+\tau)\rangle dt'. \tag{31}$$

To calculate the integral in this expression we set the correlation function of $\xi(t)$ as

$$\langle\xi(t)\xi(t+\tau)\rangle=\sigma^2e^{-\alpha\tau}\cos\omega\tau,$$

where $\sigma^2=\alpha\kappa(\omega)/2$ is the variance of the random process $\xi(t)$, $\kappa(\omega)$ is the power spectrum density of this process at the central frequency, and α is the half-width of the power spectrum of $\xi(t)$. When $1\ll\alpha\ll\omega$ we find from (31) that

$$\langle\xi(t)\delta\varphi\rangle=\frac{\sigma^2}{\omega^2}\sin\langle\varphi(t)\rangle. \tag{32}$$

Substituting (32) into (27) we obtain for $\langle \varphi(t) \rangle$ the following equation:

$$\langle \ddot{\varphi} \rangle + 2\beta\langle \dot{\varphi} \rangle + \sin\langle \varphi \rangle + \frac{\sigma^2}{2\omega^2}\sin 2\langle \varphi \rangle = 0. \qquad (33)$$

It follows herefrom that the deviation of the pendulum from its upper equilibrium position is damped out on the average; i.e., the equilibrium position is stable if

$$\sigma^2 \geqslant \omega^2. \qquad (34)$$

The transition to the regime when the pendulum's upper equilibrium position becomes stable because of both periodic and chaotic high-frequency vibration of the suspension axis can be considered as birth in the corresponding phase space of a certain attractor induced by this high-frequency vibration. In a sense we can forget this vibration and consider a new dynamical system having two stable equilibrium positions. This is precisely the approach which is developed by Blekhman in his book [28]. The technique for the derivation of the equations describing this new system is also given in this book. On the other hand, the stabilization of the pendulum's upper equilibrium position owing to random high-frequency vibration of the suspension axis, such as the excitation of the pendulum's oscillations considered above, can be treated as a certain noise-induced phase transition of the second kind, for which the parameter σ^2 plays the role of ''temperature'' and the mean frequency of the oscillations relative to the upper equilibrium position plays the role of the parameter of order. It is seen from Eq. (33) that the corresponding critical index is equal to 1/2.

VII. CONCLUSIONS

We have shown that nonequilibrium phase transitions of the second kind resulting in the appearance of an induced attractor having a certain finite dimension are possible under the influence of multiplicative noise even in such simple systems as a pendulum. Using the Rytov-Dimentberg criterion allows us to distinguish low-dimensional deterministic chaos from noise-induced oscillations, which turns out to be impossible based on correlation dimensions of the corresponding attractor. The employment of the Klimontovich criterion makes it possible to prove that, as a result of such a phase transition, the system's state becomes more ordered. There is no question that the study of noise-induced phase transitions in more complicated systems is of great physical interest. Particularly such, the study can be expected to be very useful for the elucidation of the origin of turbulence. We plan to discuss this problem in subsequent papers.

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